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Dynamic computing rough approximations approach to time-evolving information granule interval-valued ordered information system☆

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ABSTRACT

With the advent of Big Data era has seen both the volumes and update rates of data increase rapidly. The granular structure of an information system is evolving with time when redundancy data leaves and new data arrives. In order to quickly achieve the rough approximations of dynamic attribute set interval-valued ordered information system that the attribute set varies over time. In this study, we proposed two dynamic computing rough approximations approaches for time-evolving information granule interval-valued ordered information system which induced by the deletion or addition some attributes, respectively. The updating mechanisms enable obtaining additional knowledge from the varied data without forgetting the prior knowledge. According to these established computing rules, two corresponding dynamic computing algorithms are designed and some examples are illustrated to explain updating principles and show computing process. Furthermore, a series of experiments were conducted to evaluate the computational efficiency of the studied updating mechanisms based on several UCI datasets. The experimental results clearly indicate that these methods significantly outperform the traditional approaches with a dramatic reduction in the computational efficiency to update the rough approximations.

1. Introduction

In the past decades, the applications based on WEB have achieved vast development with the popularization and development of Internet/Intranet technology. The amount of freely available, user-generated data has reached an unprecedented volume. A number of data mining and information processing techniques have been proposed to capitalize on the opportunities offered by massive amount of data. The concept of Big Data, which was first identified in a *Nature* article in September 2008 [1], usually refers to massive, high-speed, and diverse information resources. It is generated by every digital process, social media exchange and almost everything around us at all times, and transmitted by all systems like sensors and mobile devices. A data set is alterable with the redundant information removes and new information arrives continuously with time in real-world applications. Dynamic computing is an efficient and rapid method for data mining in a time-evolving database, which enables acquiring additional knowledge from new data without forgetting prior knowledge [25].

Rough set theory (RST), which was first proposed by Pawlak in 1980s [28], is a valid mathematical tool for knowledge discovery and approximate reasoning. Utilizing a known knowledge in the base of knowledge to approximate characterize inaccurate and indeterminate concept is the main idea of this theory [29,30]. It is built on the basis of the classification mechanism and classified by an indiscernibility relation (equivalence relation) in a specific nonempty and finite universe [23,36]. This soft computing methodology has received great attention in recent years, and its effectiveness has been confirmed successful applications such as conflict analysis, pattern recognition, decision support and so on [14,38]. Granular Computing (GrC), another novel concept of information processing based on Zadeh's "information granularity" (an important component of artificial intelligence), which is a term of theories and techniques that makes use of granules in the process of data mining [52,53]. It identifies the essential commonalities between the surprisingly diversified problems and technologies used in applications, which could be cast into a unified framework known as a granular world [54]. The outcome of GrC is achieved

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through the interaction of information granules and the external world at a granular or numeric level by collecting the necessary information granules [42]. Where an information granule is a clump of objects that drawn together by their binary relation or proximity of functionality [31,32]. Pedrycz et al. investigated the characterization of numeric data by using a collection of information granules, enabling the key structure, topology and essential relationships of data to be described as a family of fuzzy sets [34], and formulated a general information granules framework for the description of data [35].

Due to the uncertainty of human cognitive and widely random factors exist in data where collected from practical applications. The interval-valued information system (IvIS) as an extended model of single-valued information system that attribute values being interval-valued, which is perhaps more appropriate for describing real uncertainty data. It is one of the useful ways for characterizing the values of a variable with uncertainty that utilizing the interval-valued specified by the properly defined lower and upper limits of the values that this variable possibly takes [59]. In recent years, a series of studies have been researched in the context of IvIS based on a possible degree between two interval numbers [12,43]. Dai et al. combined this concept and an extended conditional entropy to research uncertainty measurement problem in the context of interval-valued decision information system [11]. To compare with the existing rough set models, Yamaguchi utilized the grey system theory establish a novel rough set model for the information systems that containing interval data [45]. As a counterpart of the interval-number algebra, Yao detailed introduced an interval-set algebra for representing qualitative information [47], and made a comprehensive review on interval sets and interval-set algebras [49]. The dominance-based rough set approach (DRSA) is an another generalized model of the classical RST which may handle information with preference-ordered attribute domain [20]. Based on DRSA, Qian defined the interval-valued ordered information system (IvOIS) and proposed an approach of object-ranking by whole dominance degree of each object and an attribute reduction method be investigated to extract compact dominance rules [36].

In real applications, the data is generated and collected dynamically in an information system, and the knowledge discovery based on RST need to be updated accordingly [33]. The mechanism of dynamic updating approach is that dealing with the alterant data set and without re-implementing the original data mining [16,41]. It have been received much attention in past decades and been used to solve the issues that datasets are time evolving [2]. Since the form of data is represented as an attribute-value table that consisting of objects (rows) and attributes (columns) in rough set analysis [16], therefore the variation of information granules related to knowledge acquirement can be divided into three aspects and Li systematically summarized them in literature [18]. They are variation of the object set, attribute set and attribute values (means coarsening or refining of the attribute value), respectively. Recently, some excellent incremental methods for updating knowledge while the variation of the object set based on classical rough set were reported in [2,61], and a lot of noticeable incremental methods were studied for generalized rough set model. For example, Liu presented an incremental approach for inducing knowledge and learning optimization on knowledge discovery in dynamic information systems [21,22], Chen researched an incremental approach for updating rough approximations in variable precision rough set (VPRS) while objects is dynamically alter [8], and the incremental algorithm for attribute reduction of VPRS also be investigated in [5]. The incremental induction of decision rules and the selection of most interesting representatives based on dominance relation in the final group of rules were studied by Jerzy in [3], and Greco investigated them in the context of multiple criteria decision analysis [13]. In addition to these, there are many research achievements about the variation of object set [15,19,26,56,58]. On the other hand, Chan firstly presented an incremental algorithm for learning classification rules when an attribute set evolves over time [4]. Luo et al. proposed an effective method of maintaining approximations dynamically in set-valued ordered decision system under the generalization of attribute set [25], and there are a large number of researches about variation of attribute set [10,17,20,25,57,60]. Additionally, the refining or coarsening of information granule as a special case on the variation of the attribute values was firstly defined in [6]. Chen conducted a series of studies that gave an incremental algorithm for updating rough approximations in Pawlak rough set model and DRSA in incomplete information systems [7], presented matrix-based incremental algorithms for updating decision rules [9]. These research achievements provide a abundant theoretical basis for studying knowledge discovery in dynamic datasets.

However, to the best of our knowledge, the previous researches about dynamic computing rough approximations mainly concerned in the classical information systems and some generalized rough set models based on the expansion of attribute value set (e.g. set-valued information system [57], interval-valued information system [60], and hybrid information system [55]), but little attention has been paid to dynamic computing rough approximations of the IvOIS while the information granular structure varies in time. Since the interval-valued is more appropriate to describe the uncertainty of data which caused by human cognitive and widely random factors. Meanwhile, we often encounter the scenario where the ordering of properties of information system and considering attributes with preference-ordered domains is an important characteristic of multi-attribute decision making problems in practice. In order to combine these two aspects, Qian established a rough set approach in IvOIS to study the IvIS based on dominance relation [36]. Prior to this, we have studied the dynamic updating rough approximations in dynamic object set IvOIS [50]. But, there is no dynamic method for computing rough approximations of time-evolving information granule IvOIS and the approaches for computing rough approximations in other information systems cannot be utilized directly to the IvOIS. For this reason, the method of dynamic computing rough approximations in time-evolving information granule IvOIS is investigated in this paper. We discussed the principles of dynamic computing rough approximations when the attribute set varies with time and designed two dynamic computing algorithms for the variation of granular structure based on the proposed mechanism in IvOIS. Furthermore, the performances of proposed algorithms are evaluated on several varieties of UCI datasets.

The remainder of this paper is organized as follows. Some necessary preliminary knowledge of RST and IvOIS are simply introduced in Section 2. In Section 3, the approaches of dynamic computing upper and lower rough approximations in interval-valued ordered information system when the information granule is dynamic with time, and some examples are conducted to show the proposed computing mechanisms. Furthermore, two corresponding dynamic algorithms for computing rough approximations are designed for deleting and inserting attributes based on the proposed principles, respectively. The performance evaluations are conducted and the experiment results have exhibited in Section 4. Some concluding comments are offered in Section 5.

2. Preliminaries

In this section, we first briefly review some basic concepts of RST [24,28,37,39,40], and the necessary knowledge of IvOIS are introduced [36,44,46,48]. Throughout this paper, U is a finite non-empty set of objects (the universe of discourse), AT is a finite non-empty set of attributes, V is a set of attribute values, $\mathcal{P}(U)$ represents the power set of U, R_A and $IND(A)$ are indiscernibility relation, $G_{R_A}(x_i)$ is a basic granule with respect to R_A, $R_A^{\succcurlyeq}$ and $R_A^{\preccurlyeq}$ are dominance relation and dominated relation with respect to attribute set A.

2.1. Pawlak rough sets

The RST theory is built based on an information system which is a quadruple $I=(U, AT, V, f)$, where $U=\{x_1, x_2, \ldots, x_n\}$. For every attribute $a \in AT$, a set of values V_a is associated with the function $f: U \times AT \to V$ such that $f(x, a) \subseteq V_a$ for every $a \in AT$, $x \in U$. For any attribute set $A \subseteq AT$, there is an associated indiscernibility relation R_A that is defined as

$$IND(A) = \{(x, y) \in U \times U \mid \forall a \in A, f_a(x) = f_a(y)\} = R_A.$$

The indiscernibility relation R_A also named equivalence relation and divides the universe U into disjoint subsets. Such a partition is a quotient set of U, and is denoted by $U/R_A = \{[x]_{R_A} | x \in U\}$, where $[x]_{R_A} = \{y \in U | (x, y) \in R_A\}$ is the equivalence class containing x with respect to R_A, also called the Pawlak information granule [24,40].

In view of GrC, U/R_A is a granular structure that can be represented by $K(R_A) = \{G_{R_A}(x_1), G_{R_A}(x_2), \ldots, G_{R_A}(x_n)\}$, the $G_{R_A}(x_i)$ is a basic granule. Thus, a binary indiscernibility relation R_A is regarded as a granulation method for partitioning objects [37,39]. In particular, the finest granular structure on U is denoted as $K(\delta)=\{\{x_1\}, \{x_2\}, \ldots, \{x_n\}\}$, and the coarsest is denoted as $K(\omega)=\{\{x_1, x_2, \ldots, x_n\}\}$. Based on an information system, Pawlak proposed the rough set theory [28], for any $X \in \mathcal{P}(U)$ representing a basic concept and an indiscernibility relation R which induced by an attribute set A (where $A \subseteq AT$), one can respectively characterize the upper and lower approximations of X with respect to R_A by a pair of operators which be defined by following ways.

$$\bar{R_A}(X) = \{x \in U | [x]_{R_A} \cap X \neq \emptyset\},$$

$$\underline{R_A}(X) = \{x \in U | [x]_{R_A} \subseteq X\}.$$

Based on these approximation operators, other rough regions can be obtained as $pos(X) = \underline{R_A}(X)$, $neg(X) = \sim\bar{R_A}(X)$, $bn(X) = \bar{R_A}(X) - \underline{R_A}(X)$ are the positive region, negative region, and boundary region of X with respect to R_A, respectively.

2.2. Interval-valued ordered information system

An IvIS is a generalized information system that for any $x \in U$ and $a \in AT$ the $f(x, a)$ is a interval number and denoted by

$$f(x, a) = [a^L(x), a^U(x)] = \{p | a^L(x) \le p \le a^U(x); a^L(x), a^U(x) \in \mathbf{R}\}.$$

In particular, $f(x, a)$ would degenerate into a real number if $a^L(x)=a^U(x)$. Under this consideration, we regard a single-valued information system as a special form of IvIS [48]. In practical decision-making analysis, we always consider a binary dominance relation between objects that are possibly dominant in terms of value of an attribute set in an IvIS [46].

An IvIS is called IvOIS if all attributes are criterions [36], it is assumed that the domain of a criterion $a \in AT$ is completely preordered by an outranking relation $\succcurlyeq_a$ and $x \succcurlyeq_a y$ means that x is at least as good as y with respect to the criterion a. For a subset of attribute $A \subseteq AT$, we define $x \succcurlyeq_A y$ means for any $a \in A$, $x \succcurlyeq_a y$. In other words, x is at least as good as y with respect to all attributes in A. In the following, we introduce a dominance relation that identifies dominance classes to an IvOIS. In a given IvOIS, we say that x dominates y with respect to $A \subseteq AT$ if $x \succcurlyeq_A y$, and denoted by $xR_A^{\succcurlyeq}y$, that is $R_A^{\succcurlyeq} = \{(y, x) \in U \times U | y \succcurlyeq_A x\}$. It means that if $(x, y) \in R_A^{\succcurlyeq}$ then y dominates x with respect to A. That is to say, y may have a better property than x with respect to A in reality. In a similar way, the relation $R_A^{\preccurlyeq}$ (called a dominated relation) can be defined as $R_A^{\preccurlyeq} = \{(y, x) \in U \times U | x \succcurlyeq_A y\}$.

For any $A \subseteq AT$ and $A=A_1 \cup A_2$, if the attributes set A_1 according to increasing preference and A_2 according to decreasing preference, then the two binary relations can be defined more precisely as follows:

$$\begin{aligned} R_A^{\succcurlyeq} &= \{(y, x) \in U \times U | a_1^L(y) \succcurlyeq a_1^L(x), a_1^U(y) \succcurlyeq a_1^U(x), \forall a_1 \in A_1; a_2^L(y) \\ &\quad \preccurlyeq a_2^L(x), a_2^U(y) \preccurlyeq a_2^U(x), \forall a_2 \in A_2\} \\ &= \{(y, x) \in U \times U | (y, x) \in R_A^{\succcurlyeq}\}; \\ R_A^{\preccurlyeq} &= \{(y, x) \in U \times U | a_1^L(y) \preccurlyeq a_1^L(x), a_1^U(y) \preccurlyeq a_1^U(x), \forall a_1 \in A_1; \\ &\quad a_2^L(y) \succcurlyeq a_2^L(x), a_2^U(y) \succcurlyeq a_2^U(x), \forall a_2 \in A_2\} \\ &= \{(y, x) \in U \times U | (y, x) \in R_A^{\preccurlyeq}\}. \end{aligned}$$

Let $I^{\succcurlyeq}=(U, AT, V, f)$ be an IvOIS and for any $A \subseteq AT$, from the above definitions of $R_A^{\succcurlyeq}$ and $R_A^{\preccurlyeq}$, the following properties can be easily obtained that $R_A^{\succcurlyeq} = \bigcap_{a \in A} R_{\{a\}}^{\succcurlyeq}$ and $R_A^{\preccurlyeq} = \bigcap_{a \in A} R_{\{a\}}^{\preccurlyeq}$, and they are reflexive, asymmetric and transitive. The dominance class which induced by the dominance relation $R_A^{\succcurlyeq}$ is the set of objects dominating x, that is $[x]_{R_A}^{\succcurlyeq} = \{a_i^L(y) \succcurlyeq a_i^L(x),\ a_i^U(y) \succcurlyeq a_i^U(x),\ \forall a_i \in A_1;\ a_j^L(y) \preccurlyeq a_j^L(x), a_j^U(y) \preccurlyeq a_j^U(x),\ \forall a_j \in A_2\}$ and the set of objects dominated by x as $[x]_{R_A}^{\preccurlyeq} = \{a_i^L(y) \preccurlyeq a_i^L(x),\ a_i^U(y) \preccurlyeq a_i^U(x),\ \forall a_i \in A_1;\ a_j^L(y) \succcurlyeq a_j^L(x),\ a_j^U(y) \succcurlyeq a_j^U(x),\ \forall a_j \in A_2\}$. Where $[x]_A^{\succcurlyeq}$ describes the set of objects that may dominates x and $[x]_A^{\preccurlyeq}$ describes the set of objects that may dominated by x in terms of A in an IvOIS, which are called the A-dominating set and the A-dominated set with respect to $x \in U$, respectively. There is a significant lemma between dominance classless with respect to increasing preference attributes and it outlined by following way.

Lemma 2.1. [44] *Let $I^{\succcurlyeq}=(U, AT, V, f)$ be an IvOIS, for any $A_1, A_2 \subseteq AT$ and $x \in U$, then $[x]_{R_{A_1 \cup A_2}}^{\succcurlyeq} \subseteq [x]_{R_{A_1}}^{\succcurlyeq}$, $[x]_{R_{A_1 \cup A_2}}^{\succcurlyeq} \subseteq [x]_{R_{A_2}}^{\succcurlyeq}$ and $[x]_{R_{A_1 \cup A_2}}^{\succcurlyeq} = [x]_{R_{A_1}}^{\succcurlyeq} \cap [x]_{R_{A_2}}^{\succcurlyeq}$ hold.*

This lemma indicates that the more substantial attribute means the more refined of information granule. In many real application fields, one can also define the dominance relation on the universe with interval values through using other ways, the more details can be found in [44]. Furthermore, since no matter which dominance relation can be obtained similar to $R_A^{\succcurlyeq}$. Therefore, we just only adopt the dominance relation $R_A^{\succcurlyeq}$ for studying IvOIS in this paper. For simplicity and without any loss of generality we only consider attributes with increasing preference in the following investigation.

3. Dynamic computing approximations in time-evolving information granule IvOIS

Nowadays, data increase at an unprecedented rate that shows the granular structure of a given information system is evolving when the attribute set may varies with time. The dynamic computing approximations approach is an incremental method and has been proved as an useful technique for data mining in a dynamic data set, which enables acquiring additional knowledge from varied data without forgetting prior knowledge. In this section, we will research the mechanism of dynamic computing approximations in an IvOIS with time-evolving information granule which caused by the deletion or addition of attributes. The symbols Δ^- and Δ_- represent the variation of upper and lower approximations while attributes are deleted, Δ^+ and Δ_+ indicate the variation of upper and lower approximations when attributes are added, respectively.

At first, there are two significant propositions about the upper and lower approximations in IvOIS should be described.

3.1. Two significant propositions

To characterize the relationship between the rough approximations which induced by the variation of attributes, two significant propositions are studied and an example is illustrated in this subsection.

Proposition 3.1.1. *Let $I^{\succcurlyeq}=(U, AT, V, f)$ be an IvOIS, for any $A_1, A_2 \subseteq AT$ and $A_1 \subseteq A_2$, for any $X \in \mathcal{P}(U)$, the following properties hold.*

(1) $\overline{R^{\succcurlyeq}_{A_2-A_1}}(X) \supseteq \overline{R^{\succcurlyeq}_{A_2}}(X)$;
(2) $\underline{R^{\succcurlyeq}_{A_2-A_1}}(X) \subseteq \underline{R^{\succcurlyeq}_{A_2}}(X)$.

Proof.

(1) For any $A_1 \subseteq A_2 \subseteq AT$ and $x \in \overline{R^{\succcurlyeq}_{A_2}}(X)$, we can achieve that $[x]^{\succcurlyeq}_{R_{A_2}} \cap X \neq \emptyset$ based on the definition of upper approximation. According to Lemma 2.1 we can easily get that $[x]^{\succcurlyeq}_{R_{A_2}} \subseteq [x]^{\succcurlyeq}_{R_{A_2-A_1}}$, so $[x]^{\succcurlyeq}_{R_{A_2-A_1}} \cap X \neq \emptyset$ that means $x \in \overline{R^{\succcurlyeq}_{A_2-A_1}}(X)$. Therefore, we can obtain $\overline{R^{\succcurlyeq}_{A_2-A_1}}(X) \supseteq \overline{R^{\succcurlyeq}_{A_2}}(X)$.
(2) For any $x \in \underline{R^{\succcurlyeq}_{A_2-A_1}}(X)$, according to the definition of lower approximation we can get that $[x]^{\succcurlyeq}_{R_{A_2-A_1}} \subseteq X$. Furthermore, we can get $[x]^{\succcurlyeq}_{R_{A_2}} \subseteq [x]^{\succcurlyeq}_{R_{A_2-A_1}}$, so we can obtain that $[x]^{\succcurlyeq}_{R_{A_2}} \subseteq X$, namely, $x \in \underline{R^{\succcurlyeq}_{A_2}}(X)$. Thus, the proposition $\underline{R^{\succcurlyeq}_{A_2-A_1}}(X) \subseteq \underline{R^{\succcurlyeq}_{A_2}}(X)$ is proved.

Thus, the proof is finished. □

Proposition 3.1.2. *Let $I^{\succcurlyeq}=(U, AT, V, f)$ be an IvOIS, for any $A_1, A_2 \subseteq AT$ and for any $X \in \mathcal{P}(U)$, the following properties hold.*

(1) $\overline{R^{\succcurlyeq}_{A_1}}(X) \cap \overline{R^{\succcurlyeq}_{A_2}}(X) \supseteq \overline{R^{\succcurlyeq}_{A_1 \cup A_2}}(X)$;
(2) $\underline{R^{\succcurlyeq}_{A_1}}(X) \cup \underline{R^{\succcurlyeq}_{A_2}}(X) \subseteq \underline{R^{\succcurlyeq}_{A_1 \cup A_2}}(X)$.

Proof.

(1) According to the definition of the upper approximation, one can achieve that for any $x \in \overline{R^{\succcurlyeq}_{A_1 \cup A_2}}(X)$ have $[x]^{\succcurlyeq}_{R_{A_1 \cup A_2}} \cap X \neq \emptyset$. Furthermore, one can get that $[x]^{\succcurlyeq}_{R_{A_1}} \cap X \neq \emptyset$ and $[x]^{\succcurlyeq}_{R_{A_2}} \cap X \neq \emptyset$ based on Lemma 2.1. That means the object x belong to the two upper approximations, namely, $x \in \overline{R^{\succcurlyeq}_{A_1}}(X)$ and $x \in \overline{R^{\succcurlyeq}_{A_2}}(X)$. That is $x \in \overline{R^{\succcurlyeq}_{A_1}}(X) \cap \overline{R^{\succcurlyeq}_{A_2}}(X)$, to summarize, we have $\overline{R^{\succcurlyeq}_{A_1 \cup A_2}}(X) \subseteq \overline{R^{\succcurlyeq}_{A_1}}(X) \cap \overline{R^{\succcurlyeq}_{A_2}}(X)$.
(2) For any $x \in \underline{R^{\succcurlyeq}_{A_1}}(X) \cup \underline{R^{\succcurlyeq}_{A_2}}(X)$, we can obtain that $x \in \underline{R^{\succcurlyeq}_{A_1}}(X) \vee x \in \underline{R^{\succcurlyeq}_{A_2}}(X)$. According to the definition of lower approximation, one can get that $[x]^{\succcurlyeq}_{R_{A_1}} \subseteq X \vee [x]^{\succcurlyeq}_{R_{A_2}} \subseteq X$. Then, we can know that $[x]^{\succcurlyeq}_{R_{A_1 \cup A_2}} \subseteq X$, that is, $x \in \underline{R^{\succcurlyeq}_{A_1 \cup A_2}}(X)$. To sum up the above arguments, we can achieve that $\underline{R^{\succcurlyeq}_{A_1}}(X) \cup \underline{R^{\succcurlyeq}_{A_2}}(X) \subseteq \underline{R^{\succcurlyeq}_{A_1 \cup A_2}}(X)$.

Thus, the proof is accomplished.□

In order to clearly show the propositions and demonstrate the mechanism of dynamic computing approximations in IvOIS with time-evolving information granule which induced by the variation of attribute set. We illustrated an example and the data is modified from our prior study [51].

Table 1
An interval-valued ordered information system.

U	AST	LDH	α-HBDH	CK	CKMB
x_1	[10, 40]	[100, 240]	[105, 195]	[5, 195]	[0, 24]
x_2	[10, 30]	[80, 210]	[80, 180]	[10, 190]	[0, 24]
x_3	[12, 45]	[105, 248]	[100, 210]	[7, 203]	[0, 23]
x_4	[5, 30]	[60, 80]	[90, 160]	[0, 180]	[0, 10]
x_5	[10, 46]	[110, 246]	[105, 195]	[6, 198]	[0, 26]
x_6	[10, 30]	[90, 200]	[96, 206]	[5, 195]	[3, 24]
x_7	[13, 60]	[100, 240]	[115, 200]	[20,260]	[5, 30]
x_8	[10, 50]	[120, 260]	[115, 210]	[8, 196]	[5, 28]
x_9	[16, 80]	[140, 260]	[102, 300]	[40, 320]	[10, 60]
x_{10}	[8, 32]	[60, 196]	[80, 178]	[6, 160]	[2, 20]

Example 3.1. An IvOIS is presented in Table 1. It is a case of the diagnosis of myocardial infarction, where $U=\{x_1, x_2, \ldots, x_{10}\}$ representatives of ten patients and $AT=\{a_1, a_2, \ldots, a_5\}$ representatives of several enzymes related to the diagnosis of myocardial infarction. Where a_1 represents aspartate amino transferase(AST), a_2 represents Lactate dehydrogenase(LDH) and isoenzyme, a_3 represents Alfa hydroxybutyrate dehydrogenase($\alpha-$HBDH), a_4 represents Creatine Kinase(CK), a_5 represents Creatine Kinase isoenzymes(CKMB) and the measure units are μg/L.

In this example, we compute the rough approximations based on the approach of reference [36]. First, we calculate the classification which induced by the dominance relation $R^{\succcurlyeq}_{AT}$ are $[x_1]^{\succcurlyeq}_{R_{AT}}=\{x_1, x_5, x_7, x_8\}$, $[x_2]^{\succcurlyeq}_{R_{AT}}=\{x_2, x_7, x_9\}$, $[x_3]^{\succcurlyeq}_{R_{AT}}=\{x_3, x_9\}$, $[x_4]^{\succcurlyeq}_{R_{AT}}=\{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$, $[x_5]^{\succcurlyeq}_{R_{AT}}=\{x_5\}$, $[x_6]^{\succcurlyeq}_{R_{AT}}=\{x_6, x_8, x_9\}$, $[x_7]^{\succcurlyeq}_{R_{AT}}=\{x_7\}$, $[x_8]^{\succcurlyeq}_{R_{AT}}=\{x_8\}$, $[x_9]^{\succcurlyeq}_{R_{AT}}=\{x_9\}$, $[x_{10}]^{\succcurlyeq}_{R_{AT}}=\{x_7, x_8, x_9, x_{10}\}$, respectively. The calculation results indicate that dominance classes in $U/R^{\succcurlyeq}_{AT}$ do not constitute a partition of U in general, but constitute a covering of U. Let $X=\{x_1, x_3, x_5, x_7, x_9\}$ be a concept set, we can compute the approximations as follows

$$\overline{R^{\succcurlyeq}_{AT}}(X)=\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}\} \text{ and } \underline{R^{\succcurlyeq}_{AT}}(X)=\{x_3, x_5, x_7, x_9\}.$$

So the boundary of X with respect to $R^{\succcurlyeq}_{AT}$ is $Bn_{AT}(X)=\{x_1, x_2, x_4, x_6, x_{10}\}$. Let $A_1=\{\text{AST, LDH}\}$ and $A_2=\{\alpha-\text{HBDH, CK, CKMB}\}$ (it means $AT=A_1 \cup A_2$) then we can compute the followed results.

The covering of U with regard to $R^{\succcurlyeq}_{A_1}$ are $[x_1]^{\succcurlyeq}_{R_{A_1}}=\{x_1, x_3, x_5, x_7, x_8, x_9\}$, $[x_2]^{\succcurlyeq}_{R_{A_1}}=\{x_1, x_2, x_3, x_5, x_7, x_9\}$, $[x_3]^{\succcurlyeq}_{R_{A_1}}=\{x_3, x_9\}$, $[x_4]^{\succcurlyeq}_{R_{A_1}}=U$, $[x_5]^{\succcurlyeq}_{R_{A_1}}=\{x_5, x_8, x_9\}$, $[x_6]^{\succcurlyeq}_{R_{A_1}}=\{x_1, x_3, x_5, x_6, x_7, x_8, x_9\}$, $[x_7]^{\succcurlyeq}_{R_{A_1}}=\{x_7, x_9\}$, $[x_8]^{\succcurlyeq}_{R_{A_1}}=\{x_8, x_9\}$, $[x_9]^{\succcurlyeq}_{R_{A_1}}=\{x_9\}$, $[x_{10}]^{\succcurlyeq}_{R_{A_1}}=\{x_1, x_3, x_5, x_7, x_8, x_9, x_{10}\}$, then the upper and lower approximations of X with respect to $R^{\succcurlyeq}_{A_1}$ are computed as follows.

$$\overline{R^{\succcurlyeq}_{A_1}}(X)=U \text{ and } \underline{R^{\succcurlyeq}_{A_1}}(X)=\{x_3, x_7, x_9\}.$$

The covering of U with respect to $R^{\succcurlyeq}_{A_2}$ can be achieved as follows $[x_1]^{\succcurlyeq}_{R_{A_2}}=\{x_1, x_5, x_7, x_8\}$, $[x_2]^{\succcurlyeq}_{R_{A_2}}=\{x_2, x_7, x_9\}$, $[x_3]^{\succcurlyeq}_{R_{A_2}}=\{x_3, x_9\}$, $[x_4]^{\succcurlyeq}_{R_{A_2}}=\{x_1, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$, $[x_5]^{\succcurlyeq}_{R_{A_2}}=\{x_5, x_7\}$, $[x_6]^{\succcurlyeq}_{R_{A_2}}=\{x_5, x_6, x_8, x_9\}$, $[x_7]^{\succcurlyeq}_{R_{A_2}}=\{x_7\}$, $[x_8]^{\succcurlyeq}_{R_{A_2}}=\{x_8\}$, $[x_9]^{\succcurlyeq}_{R_{A_2}}=\{x_9\}$, $[x_{10}]^{\succcurlyeq}_{R_{A_2}}=\{x_7, x_8, x_9, x_{10}\}$, based on the definitions of approximations we can obtain follows.

$$\overline{R^{\succcurlyeq}_{A_2}}(X)=\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9, x_{10}\} \text{ and } \underline{R^{\succcurlyeq}_{A_2}}(X)=\{x_3, x_5, x_7, x_9\}.$$

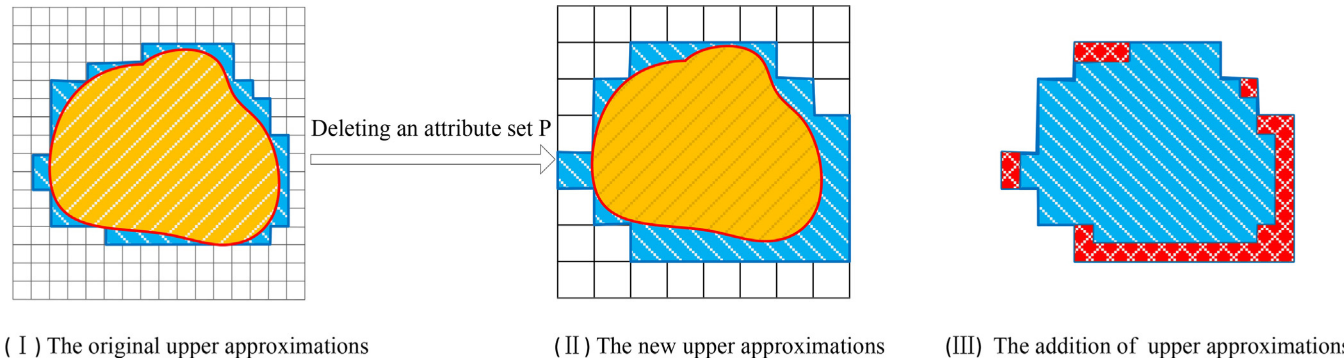

Fig. 1. The variation of upper approximation when deleting an attribute set. (For interpretation of the references to color in the text, the reader is referred to the web version of this article.)

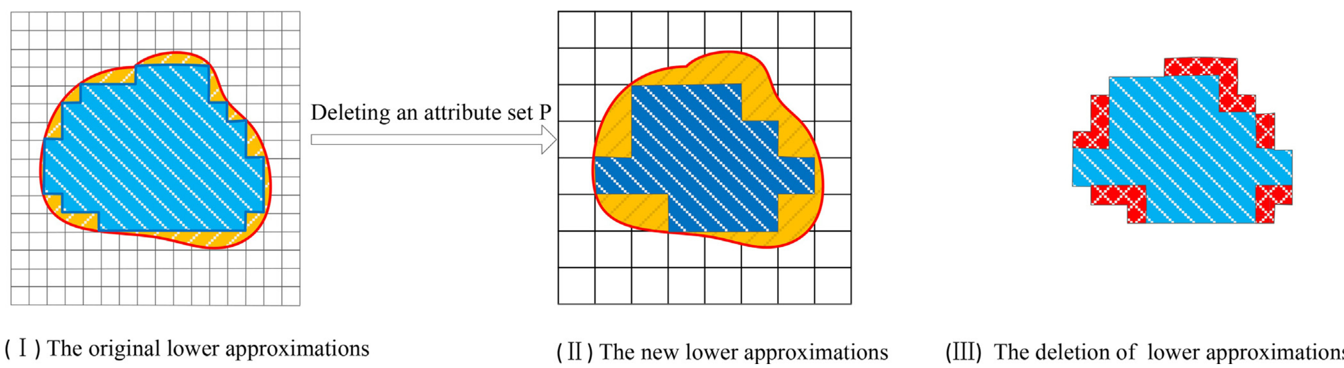

Fig. 2. The variation of lower approximation when deleting an attribute set.

According to the calculation results one can get that $\overline{R^{\succcurlyeq}_{AT-A_1}}(X) \supseteq \overline{R^{\succcurlyeq}_{AT}}(X)$, $\underline{R^{\succcurlyeq}_{AT-A_1}}(X) \subseteq \underline{R^{\succcurlyeq}_{AT}}(X)$, $\overline{R^{\succcurlyeq}_{A_1}}(X) \cap \overline{R^{\succcurlyeq}_{A_2}}(X) \supseteq \overline{R^{\succcurlyeq}_{AT}}(X)$ and $\underline{R^{\succcurlyeq}_{A_1}}(X) \cup \underline{R^{\succcurlyeq}_{A_2}}(X) \subseteq \underline{R^{\succcurlyeq}_{AT}}(X)$, respectively. They are in agreement with Lemma 2.1, Proposition 3.1.1 and Proposition 3.1.2. In the following, we will study the approaches of updating upper and lower approximations in IvOIS with time-evolving information granule which induced by the variation of attribute set and the object set remains constant. For brevity and clear, we assume the process of updating approximation have two stages, namely, from time t to time $t+1$. By considering an attribute set may enter into or get out of the given information system at time $t+1$ and we denote a dynamic IvOIS at time t as $I^{\succcurlyeq} = (U, AT, V, f)$, and at time $t+1$ the original information system be changed into $(I^{\succcurlyeq})' = (U, AT', V, f)$ with respect to insertion or deletion of some attributes.

3.2. Deletion of some attributes

In this subsection, we will investigate the approaches of computing upper and lower approximations when the deletion of some attributes (namely a attribute set P) from the original IvIOS that means $AT' = AT - P$.

Proposition 3.2.1. *Let $I^{\succcurlyeq} = (U, AT, V, f)$ be an IvOIS, for any $X \in \mathcal{P}(U)$, the upper and lower approximations of X after deleting an attribute set P can be updated as follows:*

(1) $\overline{R^{\succcurlyeq}_{AT-P}}(X) = \overline{R^{\succcurlyeq}_{AT}}(X) \cup \Delta^-$, *where* $\Delta^- = \{x \in (U - \overline{R^{\succcurlyeq}_{AT}}(X)) | [x]^{\succcurlyeq}_{R_{AT-P}} \cap X \neq \emptyset\}$;

(2) $\underline{R^{\succcurlyeq}_{AT-P}}(X) = \underline{R^{\succcurlyeq}_{AT}}(X) - \Delta_-$, *where* $\Delta_- = \{x \in \underline{R^{\succcurlyeq}_{AT}}(X) | [x]^{\succcurlyeq}_{R_{AT-P}} \not\subseteq X\}$.

Proof.

(1) According to Proposition 3.1.1, one can get that $\overline{R^{\succcurlyeq}_{AT}}(X) \subseteq \overline{R^{\succcurlyeq}_{AT-P}}(X)$ that means there is a set Δ^- such that $\overline{R^{\succcurlyeq}_{AT-P}}(X) = \overline{R^{\succcurlyeq}_{AT}}(X) \cup \Delta^-$. Furthermore we can obtain the set Δ^- is the union set which the elements not belong to $\overline{R^{\succcurlyeq}_{AT}}(X)$ before deleting but belong to $\overline{R^{\succcurlyeq}_{AT-P}}(X)$ after deleting attributes. That means the $x \in \Delta^-$ should be satisfied $x \in (U - \overline{R^{\succcurlyeq}_{AT}}(X))$ and after deleting have $[x]^{\succcurlyeq}_{R_{AT-P}} \cap X \neq \emptyset$, namely, the $\Delta^- = \{x \in (U - \overline{R^{\succcurlyeq}_{AT}}(X)) | [x]^{\succcurlyeq}_{R_{AT-P}} \cap X \neq \emptyset\}$. To more intuitive and concise display the approach of updating upper approximation, Fig. 1 is utilized to show the mechanism of dynamic computing upper approximation when an attribute set is deleted. In Fig. 1, the (I) and (II) indicate the upper approximation before and after deleting attributes. The red small block of (III) means that should be added parts and the whole (III) is the new upper approximation. It indicates that the upper approximation increase with the deletion of attributes. There should be noted that in order to let the figures looks simple and clear, we utilize a partition replace a covering of universe to show the process of variation in this investigation.

(2) Based on the definition of upper approximation and Proposition 3.1.1 (2), we know $\underline{R^{\succcurlyeq}_{AT-P}}(X) \subseteq \underline{R^{\succcurlyeq}_{AT}}(X)$. Similar to the (1) there is a set Δ_- such that $\underline{R^{\succcurlyeq}_{AT-P}}(X) = \underline{R^{\succcurlyeq}_{AT}}(X) - \Delta_-$. That means there are some elements should be deleted from the $\underline{R^{\succcurlyeq}_{AT}}(X)$ after deleting some attributes P namely the x in Δ_- are come from $\underline{R^{\succcurlyeq}_{AT}}(X)$ and there are not the lower approximations any more that means $[x]^{\succcurlyeq}_{R_{AT-P}} \not\subseteq X$. So, we can get that the $\Delta_- = \{x \in \underline{R^{\succcurlyeq}_{AT}}(X) | [x]^{\succcurlyeq}_{R_{AT-P}} \not\subseteq X\}$. Corresponding to Fig. 1, the mechanism of dynamic computing lower approximation when deleting an attribute set as shown in Fig. 2. The (I) and (II) of Fig. 2 rep-

resent the lower approximations of X with respect to AT and $AT-P$, respectively. It indicates that the lower approximation decrease with the deletion of attributes.

Thus, the proof is fulfilled.□

According to the researched mechanism of updating rough approximations when some attributes be deleted in an IvOIS, a related dynamic algorithm for updating rough approximations is designed as shown in Algorithm 1.

Algorithm 1. A dynamic algorithm for updating approximations in an IvOIS when some attributes are deleted

Input :
(1) An IvOIS $I^{\succcurlyeq}=(U,AT,V,f)$, a set $X\in\mathcal{P}(U)$ and the AT−dominating sets $[x]^{\succcurlyeq}_{R_{AT}}$ for each $x\in U$ at time t ;
(2) The original lower and upper approximations at time t : $\underline{R^{\succcurlyeq}_{AT}}(X)$, $\overline{R^{\succcurlyeq}_{AT}}(X)$;
(3) The attribute set will be deleted from $I^{\succcurlyeq}$: P.
Output : The upper and lower approximations of X in IvOIS after the deletion of P from $I^{\succcurlyeq}$ (at time $t+1$).

begin
  let : $\Delta^-=\emptyset$, $\Delta_-=\emptyset$ and $AT'=AT-P$; // initialization of the varied set ;
  compute: $U/\underline{R^{\succcurlyeq}_{AT'}}=\{[x_1]^{\succcurlyeq}_{R_{AT'}},[x_2]^{\succcurlyeq}_{R_{AT'}},\cdots,[x_{|U|}]^{\succcurlyeq}_{R_{AT'}}\}$; // calculate the covering of U with respect to $R^{\succcurlyeq}_{AT'}$;
  for $x\in(U-\underline{R^{\succcurlyeq}_{AT}}(X))$ **do**
    if $[x]^{\succcurlyeq}_{R_{AT'}}\cap X\neq\emptyset$ **then**
      $\Delta^-=\Delta^-\cup\{x\}$;
    end
  end
  $\overline{R^{\succcurlyeq}_{AT'}}(X)=\overline{R^{\succcurlyeq}_{AT}}(X)\cup\Delta^-$; // update the upper approximation of X by Proposition 3.2.1(1) ;
  for $x\in\underline{R^{\succcurlyeq}_{AT}}(X)$ **do**
    if $[x]^{\succcurlyeq}_{R_{AT'}}\not\subseteq X$ **then**
      $\Delta_-=\Delta_-\cup\{x\}$;
    end
  end
  $\underline{R^{\succcurlyeq}_{AT'}}(X)=\underline{R^{\succcurlyeq}_{AT}}(X)-\Delta_-$; // update the lower approximation of X by Proposition 3.2.1(2) ;
  return : $\underline{R^{\succcurlyeq}_{AT'}}(X)$, $\overline{R^{\succcurlyeq}_{AT'}}(X)$.
end

To illustrate the proposed propositions, we compute the upper and lower approximations of X by the classical rule and investigated method in this paper when deletion of an attribute set P, respectively.

Example 3.2. (Continued from Example 3.1) We randomly sampled $P=\{a_1,\ a_4\}$ then $AT'=\{a_2,\ a_3,\ a_5\}$, and the concept set X should be maintained that $X=\{x_1,\ x_3,\ x_5,\ x_7,\ x_9\}$. At first we calculate the approximations of X with respect to $R^{\succcurlyeq}_{AT'}$ by the definition approach and the results are represented as $\overline{R^{\succcurlyeq}_{AT'}}(X)=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_9,x_{10}\}$ and $\underline{R^{\succcurlyeq}_{AT'}}(X)=\{x_7,x_9\}$, respectively.

Based on Proposition 3.2.1, we compute the rough approximations in an IvOIS when some attributes are deleted and the computing process have been given in a step-by-step manner in accordance with Proposition 3.2.1 and Algorithm 1.

First of all, initialization of the varied set that let $\Delta^-=\emptyset$, $\Delta_-=\emptyset$ and $AT'=AT-P$.

Then, we compute the dominance class for each $x\in U$ with respect to AT' are $[x_1]^{\succcurlyeq}_{R_{AT'}}=\{x_1,x_5,x_7,x_8\}$, $[x_2]^{\succcurlyeq}_{R_{AT'}}=\{x_1,x_2,x_5,x_7,x_8,x_9\}$, $[x_3]^{\succcurlyeq}_{R_{AT'}}=\{x_3,x_8,x_9\}$, $[x_4]^{\succcurlyeq}_{R_{AT'}}=\{x_1,x_3,x_4,x_5,x_6,x_7,x_8,x_9\}$, $[x_5]^{\succcurlyeq}_{R_{AT'}}=\{x_5,x_7\}$, $[x_6]^{\succcurlyeq}_{R_{AT'}}=\{x_6,x_8,x_9\}$, $[x_7]^{\succcurlyeq}_{R_{AT'}}=\{x_7\}$, $[x_8]^{\succcurlyeq}_{R_{AT'}}=\{x_8\}$, $[x_9]^{\succcurlyeq}_{R_{AT'}}=\{x_9\}$, $[x_{10}]^{\succcurlyeq}_{R_{AT'}}=\{x_6,x_7,x_8,x_9,x_{10}\}$. So, we can get that $U-\overline{R^{\succcurlyeq}_{AT}}(X)=\{x_8\}$ and $[x_8]^{\succcurlyeq}_{R_{AT'}}=\{x_8\}\cap X=\emptyset$. That we can obtain that the $\Delta^-=\emptyset$ and $\overline{R^{\succcurlyeq}_{AT'}}(X)=\overline{R^{\succcurlyeq}_{AT}}(X)\cup\Delta^-=\overline{R^{\succcurlyeq}_{AT}}(X)\cup\emptyset=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_9,x_{10}\}$ based on Proposition 3.2.1(1).

On the other hand, for any $x\in\underline{R^{\succcurlyeq}_{AT}}(X)=\{x_3,x_5,x_7,x_9\}$, we can achieve that $[x_3]^{\succcurlyeq}_{R_{AT'}}\not\subseteq X$, $[x_5]^{\succcurlyeq}_{R_{AT'}}\not\subseteq X$, $[x_7]^{\succcurlyeq}_{R_{AT'}}\subseteq X$, $[x_9]^{\succcurlyeq}_{R_{AT'}}\subseteq X$. Based on Proposition 3.2.1(2), we can get that the $\Delta_-=\{x_3,x_5\}$ and $\underline{R^{\succcurlyeq}_{AT'}}(X)=\underline{R^{\succcurlyeq}_{AT}}(X)-\Delta_-=\underline{R^{\succcurlyeq}_{AT}}(X)-\{x_3,x_5\}=\{x_7,x_9\}$.

To summarize, according to Algorithm 1 the rough approximations of X with respect to the new attribute set AT' are listed as follows.

$$\overline{R^{\succcurlyeq}_{AT'}}(X)=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_9,x_{10}\},$$

$$\underline{R^{\succcurlyeq}_{AT'}}(X)=\{x_7,x_9\}.$$

It is obvious that the results that calculated by these two approaches are identical. However, the computational efficiency of the two methods may be different. So, we will intensive research the computational efficiency of these two mechanism that based on several UCI datasets in experimental evaluation.

3.3. Addition of some attributes

The methods of dynamic updating upper and lower approximations when inserting some attributes into the given IvIOS (that is $AT'=AT\cup Q$) will be researched in this subsection.

Proposition 3.3.1. *Let $I^{\succcurlyeq}=(U, AT, V, f)$ be an IvOIS, for any $X\in\mathcal{P}(U)$, the upper and lower approximations of X after the addition of a attribute set Q can be updated by the following way.*

(1) $\overline{R^{\succcurlyeq}_{AT\cup Q}}(X)=(\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_{Q}}(X))-\Delta^{+}$, *where* $\Delta^{+}=\{x\in(\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_{Q}}(X)-X)|[x]^{\succcurlyeq}_{R_{AT\cup Q}}\cap X=\emptyset\}$;

(2) $\underline{R^{\succcurlyeq}_{AT\cup Q}}(X)=\underline{R^{\succcurlyeq}_{AT}}(X)\cup\underline{R^{\succcurlyeq}_{Q}}(X)\cup\Delta_{+}$, *where* $\Delta_{+}=\{x\in(X-\underline{R^{\succcurlyeq}_{AT}}(X))\cap(X-\underline{R^{\succcurlyeq}_{Q}}(X))\mid[x]^{\succcurlyeq}_{R_{AT\cup Q}}\subseteq X\}$.

Proof.

(1) According to Proposition 3.1.2(1), one can easily get that $\overline{R^{\succcurlyeq}_{AT\cup Q}}(X)\subseteq\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_{Q}}(X)$. It means that there exists a set Δ^{+} such that $\overline{R^{\succcurlyeq}_{AT\cup Q}}(X)=\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_{Q}}(X)-\Delta^{+}$. And based on the definition of upper approximation we can obtain that the $X\in\overline{R^{\succcurlyeq}_{AT}}(X)$ and $X\in\overline{R^{\succcurlyeq}_{Q}}(X)$, so the elements $x\in\Delta^{+}$ are from the $\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_{Q}}(X)-X$ and should be satisfied $[x]^{\succcurlyeq}_{R_{AT\cup Q}}\cap X=\emptyset$. Thus, $\Delta^{+}=\{x\in(\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_{Q}}(X)-X)|[x]^{\succcurlyeq}_{R_{AT\cup Q}}\cap X=\emptyset\}$. The variation process of the upper approximation which induced by inserting some attributes is shown in Fig. 3. The (I) and (II) indicate the upper approximation before and after insertion attributes, respectively. The red small block means that should be deleted parts and the remainder is the new upper approximation. The (III) of Fig. 3 indicates that the upper approximation decrease with the insertion of attributes.

(1) Based on the definition of lower approximation and Proposition 3.1.2, we can get that $\underline{R^{\succcurlyeq}_{AT}}(X)\cup\underline{R^{\succcurlyeq}_{Q}}(X)\subseteq\underline{R^{\succcurlyeq}_{AT\cup Q}}(X)\subseteq X$. We can assume that there exists a set Δ_{+} such that $\underline{R^{\succcurlyeq}_{AT\cup Q}}(X)=\underline{R^{\succcurlyeq}_{AT}}(X)\cup\underline{R^{\succcurlyeq}_{Q}}(X)\cup\Delta_{+}$. Furthermore, the elements of Δ_{+} are come from $X-\underline{R^{\succcurlyeq}_{AT}}(X)$ and $X-\underline{R^{\succcurlyeq}_{Q}}(X)$, and should satisfy the definition of lower approximations that $[x]^{\succcurlyeq}_{R_{AT\cup Q}}\subseteq X\}$. Hence, we can obtain that $\Delta_{+}=\{x\in(X-\underline{R^{\succcurlyeq}_{AT}}(X))\cap(X-\underline{R^{\succcurlyeq}_{Q}}(X))|[x]^{\succcurlyeq}_{R_{AT\cup Q}}\subseteq X\}$. Fig. 4 shows the variation process of the lower approximation with respect to adding an attribute set. The (I) and (II) represent the lower approximation of X with regard to AT and $AT\cup Q$, respectively. The red small block of (III) means the parts be added into lower approximation while insertion of some attributes. It indicates that the lower approximation increase with the insertion of attributes.

Thus, the proof is fulfilled.□

Algorithm 2 is an algorithm of computing rough approximations with the addition of attributes and it is designed based on Proposition 3.3.1. It will be used to compare the proposed method and the traditional approach in the aspect of computational efficiency.

Algorithm 2. An dynamic algorithm for updating approximations in an IvOIS when some attributes are inserted

Input :
(1) An IvOIS $I^{\succcurlyeq}=(U, AT, V, f)$, a set $X\in\mathcal{P}(U)$ and the $AT-$dominating sets $[x]^{\succcurlyeq}_{R_{AT}}$ for each $x\in U$ at time t ;
(2) The original covering of U and approximations set of X at time t : $U/R^{\succcurlyeq}_{AT}$, $\underline{R^{\succcurlyeq}_{AT}}(X)$ and $\overline{R^{\succcurlyeq}_{AT}}(X)$;
(3) The attribute set will be inserted into $I^{\succcurlyeq}$: Q.
Output : The upper and lower approximations of X in IvOIS at time $t+1$ after the insertion of Q into $I^{\succcurlyeq}$.

begin
  let : $\Delta^{+}=\emptyset$, $\Delta_{+}=\emptyset$ and $AT'=AT\cup Q$; // initialization of the varied set ;
  compute : $U/R^{\succcurlyeq}_{Q}=\{[x_1]^{\succcurlyeq}_{R_Q}, [x_2]^{\succcurlyeq}_{R_Q}, \cdots, [x_{|U|}]^{\succcurlyeq}_{R_Q}\}$, and $U/R^{\succcurlyeq}_{AT'}=U/R^{\succcurlyeq}_{AT}\cap U/R^{\succcurlyeq}_{Q}$;
  compute : $\overline{R^{\succcurlyeq}_{Q}}(X)$ and $\underline{R^{\succcurlyeq}_{Q}}(X)$; // according to definitions to compute the approximations of X ;
  for $x\in(\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_{Q}}(X)-X)$ **do**
    if $[x]^{\succcurlyeq}_{R_{AT'}}\cap X=\emptyset$ **then**
      $\Delta^{+}=\Delta^{+}\cup\{x\}$;
    end
  end
  $\overline{R^{\succcurlyeq}_{AT'}}(X)=\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_{Q}}(X)-\Delta^{+}$; // update the upper approximation of X by Proposition 3.3.1(1) ;
  for $x\in((X-\underline{R^{\succcurlyeq}_{AT}}(X))\cap(X-\underline{R^{\succcurlyeq}_{AT}}(X)))$ **do**
    if $[x]^{\succcurlyeq}_{R_{AT'}}\subseteq X$ **then**
      $\Delta_{+}=\Delta_{+}\cup\{x\}$;
    end
  end
  $\underline{R^{\succcurlyeq}_{AT'}}(X)=\underline{R^{\succcurlyeq}_{AT}}(X)\cup\underline{R^{\succcurlyeq}_{Q}}(X)\cup\Delta_{+}$; // update the lower approximation of X by Proposition 3.3.1(2) ;
  return : $\underline{R^{\succcurlyeq}_{AT'}}(X)$, $\overline{R^{\succcurlyeq}_{AT'}}(X)$.
end

Example 3.3. (Continued from Example 3.1) Let $Q=\{a_6, a_7\}$ is an attribute set that will be inserted to $I^{\succcurlyeq}$ and the characteristics as shown in Table 2. The a_6 and a_7 represent Cardiac Troponin I(cTnI) and Myoglobin(MYO), respectively. Based on the definitions of rough approximations, we can compute the upper and lower approximations of X with respect to $AT'=AT\cup Q$ and the results are $\overline{R^{\succcurlyeq}_{AT'}}(X)=\{x_1, x_2, x_3, x_4, x_5, x_7, x_9, x_{10}\}$ and $\underline{R^{\succcurlyeq}_{AT'}}(X)=\{x_1, x_3, x_5, x_7, x_9\}$, respectively.

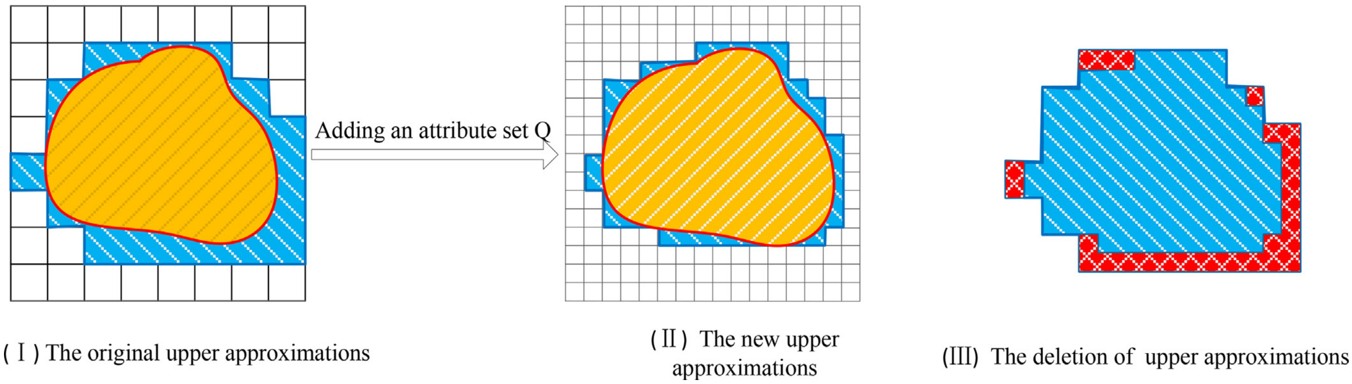

Fig. 3. The variation of upper approximations when adding an attribute set. (For interpretation of the references to color in the text, the reader is referred to the web version of this article.)

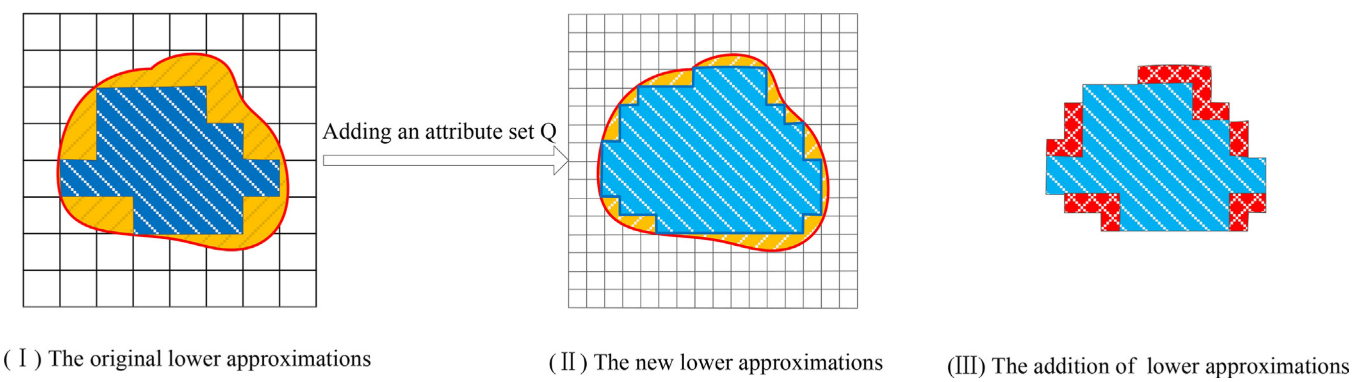

Fig. 4. The variation of lower approximations when adding an attribute set. (For interpretation of the references to color in the text, the reader is referred to the web version of this article.)

Table 2
The attribute set Q be added into $I^{\succcurlyeq}$.

U	cTnI	MYO
x_1	[0.12, 0.38]	[23, 58]
x_2	[0.09, 0.45]	[21, 63]
x_3	[0.11, 0.43]	[28, 58]
x_4	[0.10, 0.42]	[22, 57]
x_5	[0.12, 0.38]	[23, 58]
x_6	[0.14, 0.39]	[35, 65]
x_7	[0.11, 0.43]	[28, 58]
x_8	[0.13, 0.43]	[21, 63]
x_9	[0.25, 0.51]	[34, 65]
x_{10}	[0.12, 0.38]	[23, 58]

Based on Proposition 3.3.1, we compute the rough approximations in an IvOIS when some attributes are added and the computing process have been given in a step-by-step manner in accordance with Proposition 3.3.1 and Algorithm 2.

At first, initialization of the varied set that let $\Delta^+=\emptyset$, $\Delta_+=\emptyset$ and $AT'=AT\cup Q$.

Then, we compute the covering of U with respect to the added attribute set Q, they are $[x_1]^{\succcurlyeq}_{R_Q}=\{x_1,x_5,x_6,x_9,x_{10}\}$, $[x_2]^{\succcurlyeq}_{R_Q}=\{x_2,x_9\}$, $[x_3]^{\succcurlyeq}_{R_Q}=\{x_3,x_7,x_9\}$, $[x_4]^{\succcurlyeq}_{R_Q}=\{x_3,x_4,x_7,x_9\}$, $[x_5]^{\succcurlyeq}_{R_Q}=\{x_1,x_5,x_6,x_9,x_{10}\}$, $[x_6]^{\succcurlyeq}_{R_Q}=\{x_6\}$, $[x_7]^{\succcurlyeq}_{R_Q}=\{x_3,x_7,x_9\}$, $[x_8]^{\succcurlyeq}_{R_Q}=\{x_8,x_9\}$, $[x_9]^{\succcurlyeq}_{R_Q}=\{x_9\}$, $[x_{10}]^{\succcurlyeq}_{R_Q}=\{x_1,x_5,x_6,x_9,x_{10}\}$. So, we can get that the approximations of X with regard to $R^{\succcurlyeq}_Q$ based on the definitions of rough approximations, and the rough approximations are $\overline{R^{\succcurlyeq}_Q}(X)=\{x_1,x_2,x_3,x_4,x_5,x_7,x_8,x_9,x_{10}\}$ and $\underline{R^{\succcurlyeq}_Q}(X)=\{x_3,x_7,x_9\}$.

In addition, we can achieve that the covering of U with respect to AT' are $[x_1]^{\succcurlyeq}_{R_{AT'}}=\{x_1,x_5\}$, $[x_2]^{\succcurlyeq}_{R_{AT'}}=\{x_2,x_9\}$, $[x_3]^{\succcurlyeq}_{R_{AT'}}=\{x_3,x_9\}$, $[x_4]^{\succcurlyeq}_{R_{AT'}}=\{x_3,x_4,x_7,x_9\}$, $[x_5]^{\succcurlyeq}_{R_{AT'}}=\{x_5\}$, $[x_6]^{\succcurlyeq}_{R_{AT'}}=\{x_6\}$, $[x_7]^{\succcurlyeq}_{R_{AT'}}=\{x_7\}$, $[x_8]^{\succcurlyeq}_{R_{AT'}}=\{x_8\}$, $[x_9]^{\succcurlyeq}_{R_{AT'}}=\{x_9\}$, $[x_{10}]^{\succcurlyeq}_{R_{AT'}}=\{x_9,x_{10}\}$ based on Lemma 2.1.

Furthermore, according to Example 3.1 and Proposition 3.3.1(1), we can get that $\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_Q}(X)-X=\{x_1,x_2,x_3,x_4,x_5,x_7,x_9,x_{10}\}-\{x_1,x_3,x_5,x_7,x_9\}=\{x_2,x_4,x_{10}\}$, and because $[x_2]^{\succcurlyeq}_{R_{AT'}}\cap X=\{x_9\}$, $[x_4]^{\succcurlyeq}_{R_{AT'}}\cap X=\{x_3,x_7,x_9\}$, $[x_{10}]^{\succcurlyeq}_{R_{AT'}}\cap X=\{x_9\}$. So, the $\Delta^+=\emptyset$. That means that $\overline{R^{\succcurlyeq}_{AT'}}(X)=(\overline{R^{\succcurlyeq}_{AT}}(X)\cap\overline{R^{\succcurlyeq}_Q}(X))-\Delta^+=\{x_1,x_2,x_3,x_4,x_5,x_7,x_9,x_{10}\}-\emptyset=\{x_1,x_2,x_3,x_4,x_5,x_7,x_9,x_{10}\}$.

On the other hand, we can compute $(X-\overline{R^{\succcurlyeq}_{AT}}(X))\cap(X-\overline{R^{\succcurlyeq}_Q}(X))=(X-\{x_3,x_5,x_7,x_9\})\cap(X-\{x_3,x_7,x_9\})=\{x_1\}\cap\{x_1,x_5\}=\{x_1\}$, and $[x_1]^{\succcurlyeq}_{AT'}=\{x_1,x_5\}\subseteq X$ based on step 11 of Algorithm 2. So, the $\Delta_+=\{x_1\}$. According to Proposition 3.3.1(2), we can obtain that $\underline{R^{\succcurlyeq}_{AT'}}(X)=\underline{R^{\succcurlyeq}_{AT}}(X)\cup\underline{R^{\succcurlyeq}_Q}(X)\cup\Delta_+=\{x_3,x_5,x_7,x_9\}\cup\{x_3,x_7,x_9\}\cup\{x_1\}=\{x_1,x_3,x_5,x_7,x_9\}$.

To sum up the above calculated results, we can obtain that the rough approximations of X with respect to AT' after inserting an attribute set Q and they are listed as follows.

$$\overline{R^{\succcurlyeq}_{AT'}}(X)=\{x_1,x_2,x_3,x_4,x_5,x_7,x_9,x_{10}\},$$

$$\underline{R^{\succcurlyeq}_{AT'}}(X)=\{x_1,x_3,x_5,x_7,x_9\}.$$

It is easy to see that the achievements of the two methods are the same. But there may be exist differences in the computational efficiency between the two approaches. In order to test the effect

Table 3
Experiment datasets.

No.	Data set name	Abbreviation	Objects	Attributes
1	Energy efficiency	EE	768	8
2	Airfoil self-noise	AS	1503	6
3	Wine quality-red	WQ-r	1599	11
4	Wine quality-white	WQ-w	4898	11
5	Letter recognition	LR	8084	16
6	Spoken Arabic digit	SAD	8800	13

of the proposed updating mechanism, some pertinent experiments are designed in the next section.

4. Experiment evaluations

To evaluate the performance of the proposed dynamic approach, we conduct a series of experiments to compare the computational efficiency between the classical method and the proposed approach for computing approximations based on serval standard datasets from [27], which named "Energy efficiency", "Airfoil Self-Noise", "Wine Quality-red", "Wine Quality-white", "Letter Recognition", "Spoken Arabic Digit" and the characteristics of the datasets are summarized in Table 3. It should be noted that any interval-valued datasets are suitable for validating the dynamic computing rough approximation approach if the cardinality of attributes are bigger than 1. In our experimental studies, to eliminate the uncertainty that caused by datasets and provide sufficient evidence to evaluate the performance of the proposed dynamic approach, we select six UCI datasets which include different cardinality of objects and attributes. The computing program of experiment is running on a personal computer with CPU is i3-370 2.40 GHz, Windows 7(32-bit) and the program are coded by C language and platform is VC++ 6.0.

There should be noted that the attributes characteristics of the six datasets in Table 3 are real number. To carry out the experiment, we construct the interval-valued information tables by utilizing multiply error precision α, namely, the attribute value of $x_i \in U$ with respect to $a_j \in AT$ is $v_{(x_i,a_j)}$, we can let it express as $[(1-\alpha)\times V_{(x_i,a_j)}, (1+\alpha)\times V_{(x_i,a_j)}]$. In this paper, we set the error precision $\alpha = 0.05$, this construction method can refer to our prior study [50]. In the following experiments, to ensure the sufficiency and effectiveness of the experiment, we repeat 5 times for each sub-experiment and take the arithmetic mean of the 5 calculation results as the final result to remove the uncertainty of computer. The experimental results are the computational time that spent on the process of obtaining the upper and lower approximation sets in the new system. Without loss of generality, we randomly take a set as the concept set X and the attribute set should be deleted or added is variable in different rates.

4.1. Experiments of deleting attributes

In this subsection, the experiments of deleting attributes will de conducted based on Proposition 3.2.1 and Algorithm 1. We randomly take a object set $X \in \mathcal{P}(U)$ as the concept set and the cardinality of the selected set is $|X| = 30\,\% \times |U|$, set the all attributes of the given IvOIS as the initial value and delete a proportion of the attributes in each experiment, the rate is from 5% to 50% of $|AT|$ and the size of increase step is 5%. It should be noted that the attributes will be removed are same for two computational methods in one experiment. The number of the deleted attributes may be a decimal, so the Integer-valued function is used into our experiments. For example, the 5% attributes of the AS that $5\,\% \times |AT| = 5\,\% \times 6 = 0.3$ but the value of Integer-valued function is 1. The Integer-valued function is usually expressed as $[5\,\% \times |AT|]$ or $INT(5\,\% \times |AT|)$ and this way will be utilized into the throughout of these experiments. According to these setting of experiments and the designed two dynamic algorithms, we can utilize the proposed updating mechanism to get the results as shown in Table 4, the unit of the experimental result is seconds(s).

According to these experimental results, we can achieve that the proposed updating rough approximations approach is more effective than the traditional method in the aspect of computational efficiency when some attributes be deleted. In order to compare the differences between the two methods in terms of computational efficiency, we draw two trend lines to show the changes of them based on the achievements in Table 4.

The Fig. 5 consists of 6 subgraphs, they correspond to the data where from the Table 4. In each sub-figure (I)-(VI) of Fig. 5, the $x-$coordinate pertains to the ratio of the numbers of the deleted attributes and original data, while the $y-$coordinate concerns the computational time. The blue and green line mean the computational efficiency of proposed way and traditional approach, respectively. From the Fig. 5, we can know that the computational time of Algorithm 1 and the traditional method are monotone. The computational time of dynamic approach rises monotonically with the increasing number of deleted attributes and the increasing cardinality of attribute set. On the other hand, the computational time of traditional method decreases monotonically with the increasing number of deleted attributes and the increasing cardinality of attribute set. It is clear that the performance of Algorithm 1 is influenced by the numbers of delated attributes for any dataset. From the sub-figure (I) and (II), we can get that the cardinality of the experimental attribute set influences the performance of Algorithm 1 on one data set when ratio of deleted are same. It implies that the size of a dataset is one of factors influencing the performance of Algorithm 1. Furthermore, it is obviously that all of these figures indicate that the dynamic approach more effective. With more and more attributes are changed, the advantages of the proposed method may become smaller. There are still advantages in terms of computational efficiency even if the varied attribute set is bigger than the remainder attribute set. However, it is also should be noted that we may consider other more effective methods when the varied attribute set is far greater than the set of attributes of the original data. To summarize, the approach of dynamic computing rough approximations is more effective than the classical method in the viewpoint of computational time when deletion of attributes.

4.2. Experiments of adding attributes

The experiments of adding some new attributes will de carried out based on Proposition 3.3.1 and Algorithm 2 in this subsection. Similar to the previous experiments, we randomly take a object set $X \in \mathcal{P}(U)$ as the concept set and the cardinality of the selected set is $|X| = 30\,\% \times |U|$, take the 60% of the original attributes as the initial value of the experiment and the remainder attributes as the test set. The experiments is divided into 10 times, different rates attributes are inserted and the proportion is from 10% to 100% of the test set that means the size of the increase step is 10%. Based on these requirements and the researched mechanism of updating approximations, the results of this experiment as listed in the Table 5.

Table 5 shows the computation time of updating rough approximations when some attributes are added. Similar to the experiments of deleting some attributes from $I^{\succcurlyeq}$, we also adopt such schemes to compare the performance of algorithms on the case of inserting the attributes into the $I^{\succcurlyeq}$. More detailed change trend lines of these two approaches with the increasing ratio of datasets are given in Fig. 6.

In each sub-figure (I)–(VI) of Fig. 6, the x-coordinate pertains to the ratio of the numbers of the inserted attributes and test

Table 4
A comparison of definition and dynamic approach versus different updating rates when deleting attributes.

Del. (%)	EE		AS		WQ-r		WQ-w		LR		SAD	
	Def.	Dyn.	Def.	Dyn.	Def.	Dyn.	Def.	Dyn.	Def.	Dyn.	Def.	Dyn.
5%	0.087	0.012	0.223	0.022	0.571	0.096	4.671	0.709	24.725	2.452	22.634	2.195
10%	0.084	0.013	0.240	0.027	0.519	0.096	4.233	0.717	23.084	2.457	20.815	2.541
15%	0.073	0.018	0.220	0.025	0.510	0.101	4.215	0.710	21.196	2.842	20.809	2.539
20%	0.072	0.017	0.200	0.029	0.448	0.114	3.809	0.809	19.649	3.185	19.113	2.898
25%	0.073	0.018	0.192	0.038	0.466	0.110	3.781	0.813	19.642	3.494	17.005	3.247
30%	0.063	0.022	0.194	0.039	0.393	0.130	3.348	0.921	18.121	3.810	17.131	3.242
35%	0.063	0.023	0.156	0.037	0.384	0.135	3.367	0.929	16.588	3.798	15.291	3.669
40%	0.054	0.026	0.161	0.050	0.339	0.154	2.968	1.034	14.911	4.107	13.533	4.005
45%	0.054	0.027	0.158	0.052	0.352	0.143	2.961	1.040	12.978	4.487	13.515	4.007
50%	0.055	0.026	0.168	0.054	0.339	0.168	2.994	1.154	12.981	4.814	11.857	4.372

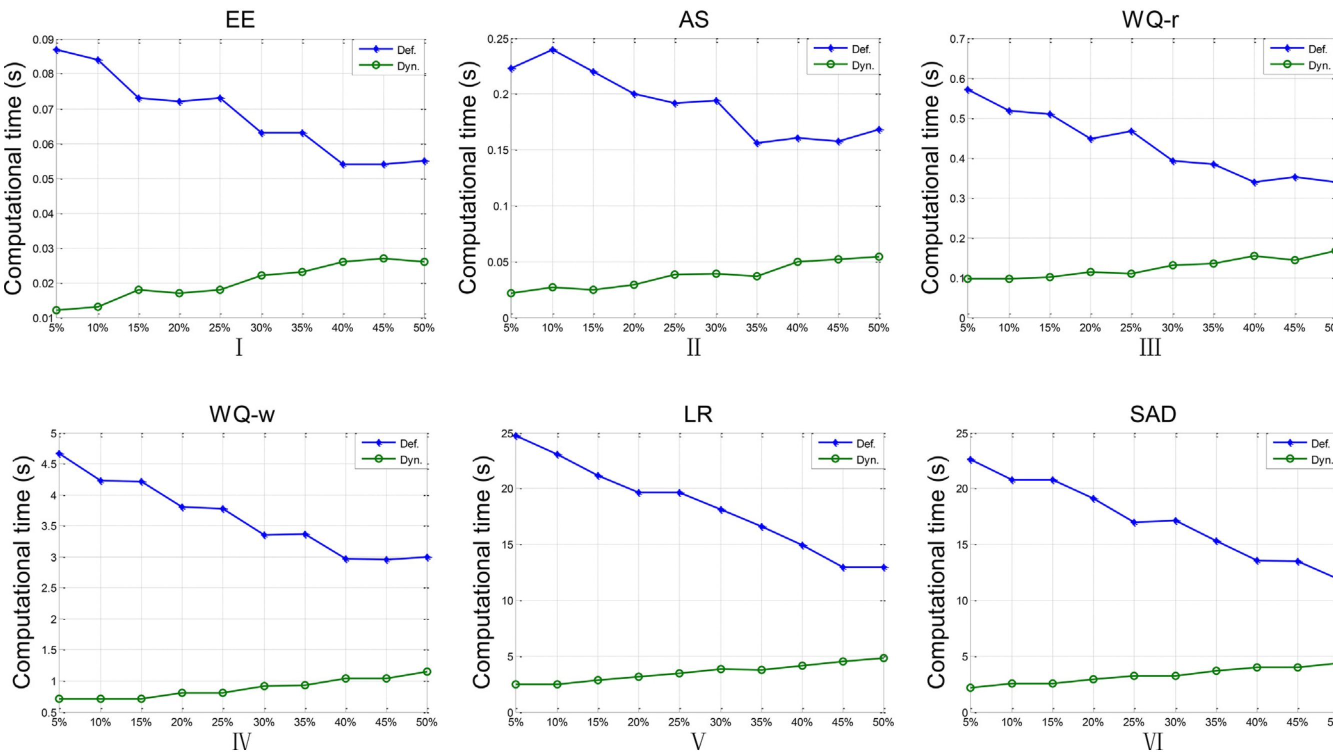

Fig. 5. A comparison of definition and dynamic approach versus different updating rates when deleting attributes. (For interpretation of the references to color in the text, the reader is referred to the web version of this article.)

Table 5
A comparison of definition and dynamic approach versus different updating rates when adding attributes.

Del.(%)	EE		AS		WQ-r		WQ-w		LR		SAD	
	Def.	Dyn.	Def.	Dyn.	Def.	Dyn.	Def.	Dyn.	Def.	Dyn.	Def.	Dyn.
10%	0.060	0.014	0.195	0.045	0.402	0.101	3.600	0.895	16.452	4.209	15.523	3.925
20%	0.061	0.015	0.196	0.046	0.387	0.095	3.610	0.899	17.948	4.612	17.309	4.406
30%	0.072	0.018	0.201	0.047	0.451	0.113	4.101	1.031	19.506	5.026	17.243	4.389
40%	0.071	0.017	0.223	0.054	0.448	0.112	4.084	1.024	19.340	4.985	19.231	4.919
50%	0.074	0.018	0.232	0.056	0.510	0.129	4.658	1.179	21.206	5.481	19.272	4.928
60%	0.086	0.021	0.229	0.055	0.514	0.130	4.730	1.193	22.930	5.941	21.006	5.392
70%	0.083	0.020	0.272	0.066	0.579	0.147	5.283	1.345	22.820	5.911	23.051	5.940
80%	0.091	0.023	0.269	0.066	0.572	0.146	5.250	1.336	24.253	6.230	22.841	5.501
90%	0.089	0.022	0.273	0.067	0.649	0.166	6.179	1.584	25.543	6.377	23.538	6.071
100%	0.091	0.023	0.283	0.068	0.653	0.167	6.198	1.589	25.539	6.586	23.648	6.021

data, while the y-coordinate concerns the computational time. The tendency chart is drawn from the data in table t6. It is similar to the deletion of attributes in the previous subsection. The blue and green line mean the computational efficiency of proposed way and traditional approach, respectively. From the sub-figure (I) to sub-figure (VI) of Fig. 6, we can achieve that the computational time of Algorithm 2 and the traditional computing method increase monotonically with the increase of the number of inserted attributes and the cardinality of attribute set. According to the sub-figure (III) and (IV), we can obtain that the performance of Algorithm 2 is influenced by the cardinality of object set while the cardinality of attribute set is constant. The greater cardinality of

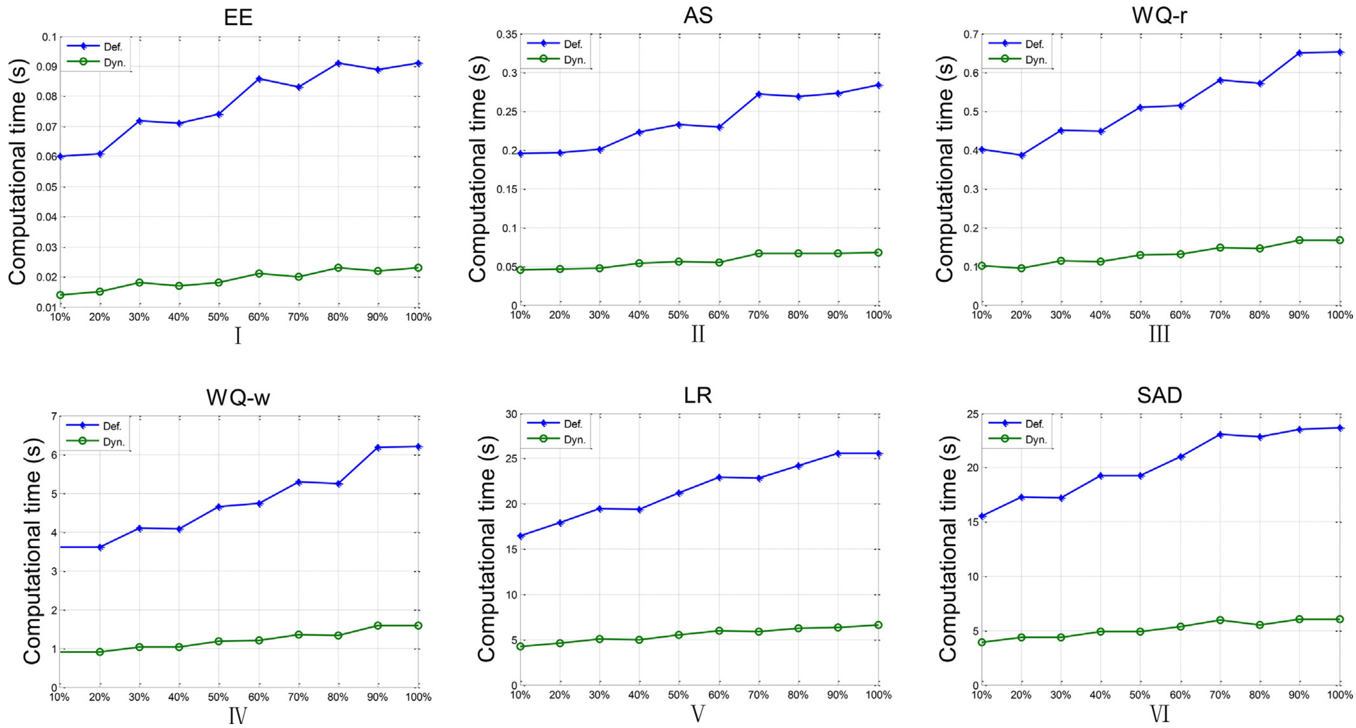

Fig. 6. A comparison of definition and dynamic approach versus different updating rates when adding attributes. (For interpretation of the references to color in the text, the reader is referred to the web version of this article.)

object set, the lower computational efficiency. Furthermore, the sub-figure (V) and (VI) imply that the computational efficiency of dynamic approach is more efficient than the classical computing way still hold when the cardinality of object set and attribute are big enough. Based on these achievements, it is easy to get the proposed mechanism always performs faster than the classical method for computing rough approximations if some attributes be inserted into the original dataset. Compared with the deletion attributes, there is difference that the proposed method is always more efficient than the traditional approach in the view of computational efficiency. To reduce the computation time and improve the efficiency, the studied rough approximations updating mechanisms can be utilized into data processing of time-evolving information granule IvIOS.

5. Conclusions

The dynamic updating rough approximations approach is an incremental method, which enables acquiring additional knowledge from the alterative data without forgetting the prior knowledge. It is a very effective approach to maintain knowledge in the time-evolving environment. In this study, we researched the mechanisms of dynamic updating upper and lower rough approximations in time-evolving information granule IvOIS, where the granular structure is varying with the deletion and addition of some attributes. Two principles of dynamic updating rough approximations are established for the coarsening or refinement of granular structure in IvOIS, respectively. Then, a series of medical diagnosis examples are illustrated to explicate the studied propositions and updating mechanisms. According to the proposed updating rules, we designed two algorithms for computing rough approximations when deleting or inserting attributes, respectively. Furthermore, a group of experiments are conducted based on six UCI datasets, and the experimental results indicate that the proposed mechanisms improve the computational efficiency for updating rough approximations when the information granular structure varies with time. In practice applications, the process of data change over time is very manifold and complicated. So, the established approach is useful to handle with the time-evolving granular structure that coarsening or refinement of information granule in IvIOS. In order to take further study with massive data, we will strive to improve the computational efficiency and storage efficiency of these methods in our future work.

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